Problem 11

A planning engineer for a new alum plant must present some estimates to his company regarding the capacity of a silo designed to contain bauxite ore until it is processed into alum. The ore resembles pink talcum powder and is poured from a conveyor at the top of the silo. The silo is a cylinder 100 ft high with a radius of 200 ft. The conveyor carries ore at a rate of $60,000\pi$ ft^3/h and the ore maintains a conical shape whose radius is 1.5 times its height.

(a) If, at a certain time t, the pile is 60 ft high, how long will it take for the pile to reach the top of the silo?

(b) Management wants to know how much room will be left in the floor area of the silo when the pile is 60 ft high. How fast is the floor area of the pile growing at that height?

(c) Suppose a loader starts removing the ore at the rate of $20,000\pi$ ft^3/h when the height of the pile reaches 90 ft. Suppose, also, that the pile continues to maintain its shape. How long will it take for the pile to reach the top of the silo under these conditions?

Solution

Part (a)

To answer the question we need to look for a relationship between the height of the pile and the time. Because of the units, cubic feet per hour, we know that the given number is volume per unit time. Choose V to represent volume and t to represent time.

$$\frac{dV}{dt} = 60,000\pi \tag{1}$$

This is how fast the volume of the cone grows. Recall that the volume of a cone is

$$V = \frac{1}{3}\pi r^2 h,$$

where r is the radius of the cone's base and h is the cone's height. We're told that the radius is 1.5 times the height, so we can substitute $r = (3/2)h$ to get the volume in terms of height only.

$$\begin{aligned} V(h) &= \frac{1}{3}\pi\left(\frac{3}{2}h\right)^2 h \\ &= \frac{3}{4}\pi h^3 \end{aligned}$$

To make use of equation (1), differentiate $V(h)$ with respect to time. We have to use the chain rule here since V is a function of h and h is a function of t.

$$\frac{dV(h)}{dt} = \frac{dV}{dh}\frac{dh}{dt}$$

Plug in $60,000\pi$ to the left side and evaluate dV/dh.

$$\frac{dV}{dh} = \frac{9}{4}\pi h^2$$

So we have

$$60,000\pi = \frac{9}{4}\pi h^2 \frac{dh}{dt}, \tag{2}$$

which is a first-order differential equation we can solve with separation of variables. Divide both sides by $(9/4)\pi$.

$$\frac{80,000}{3} = h^2 \frac{dh}{dt}$$

Separate variables.

$$\frac{80,000}{3}\, dt = h^2\, dh$$

Integrate both sides.

$$\frac{80,000}{3} t + C = \frac{1}{3} h^3$$

Multiply both sides by 3.

$$80,000t + 3C = h^3$$

To determine the constant of integration, we assume that the height of the pile at $t = 0$ is zero, that is, $h(0) = 0$.

$$0 + 3C = 0 \quad \to \quad C = 0$$

The equation for the height simplifies to

$$80,000t = h^3.$$

Divide both sides by $80,000$ to solve for t.

$$t(h) = \frac{h^3}{80,000}$$

We have the time as a function of height now. The question is asking how long it takes for the pile to go from 60 feet to 100 feet, so we evaluate $t(60)$ and $t(100)$ and then take the difference.

$$t(60) = \frac{60^3}{80,000} = 2.7$$
$$t(100) = \frac{100^3}{80,000} = 12.5$$

12.5 - 2.7 = 9.8. Therefore, it takes 9.8 hours for the pile to go from 60 feet to 100 feet, the top of the silo.

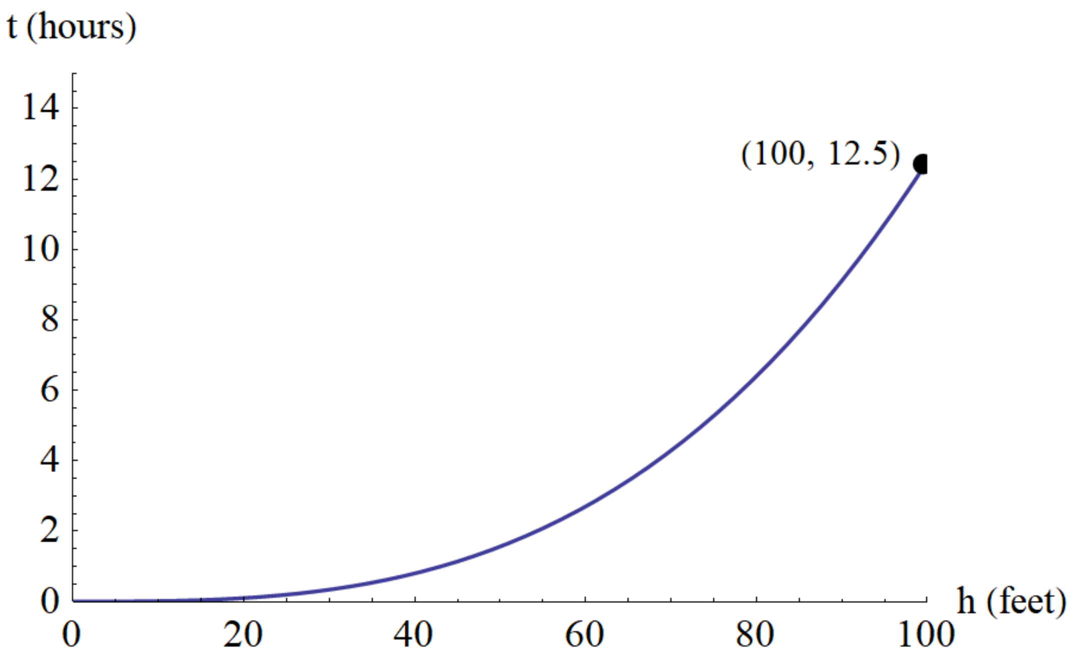

Figure 1: Plot of the time it takes for the pile to get to a certain height.

Part (b)

When the pile is 60 feet high, the radius at its base is 90 feet. The floor area left is just the area of the silo's base minus the area of the pile's base.

$$\text{Floor area left:} \quad \pi(200)^2 - \pi(90)^2 = 31,900\pi$$

The second part of the question asks for a rate of change when the pile is 60 feet high. Let the pile's area be represented as A. The word "fast" means we'll be taking a derivative of A with respect to time. we thus have to find

$$\left.\frac{dA}{dt}\right|_{h=60}.$$

The area of the pile's base is circular, so

$$A = \pi r^2.$$

Since $r = (3/2)h$, we can write the area as a function of height.

$$\begin{aligned} A(h) &= \pi\left(\frac{3}{2}h\right)^2 \\ &= \frac{9}{4}\pi h^2 \end{aligned}$$

We have to evaluate the derivative when the height is 60 feet, so take the derivative of $A(h)$ with respect to time. We have to use the chain rule here since A is a function of h and h is a function of t.

$$\frac{dA(h)}{dt} = \frac{dA}{dh}\frac{dh}{dt}$$

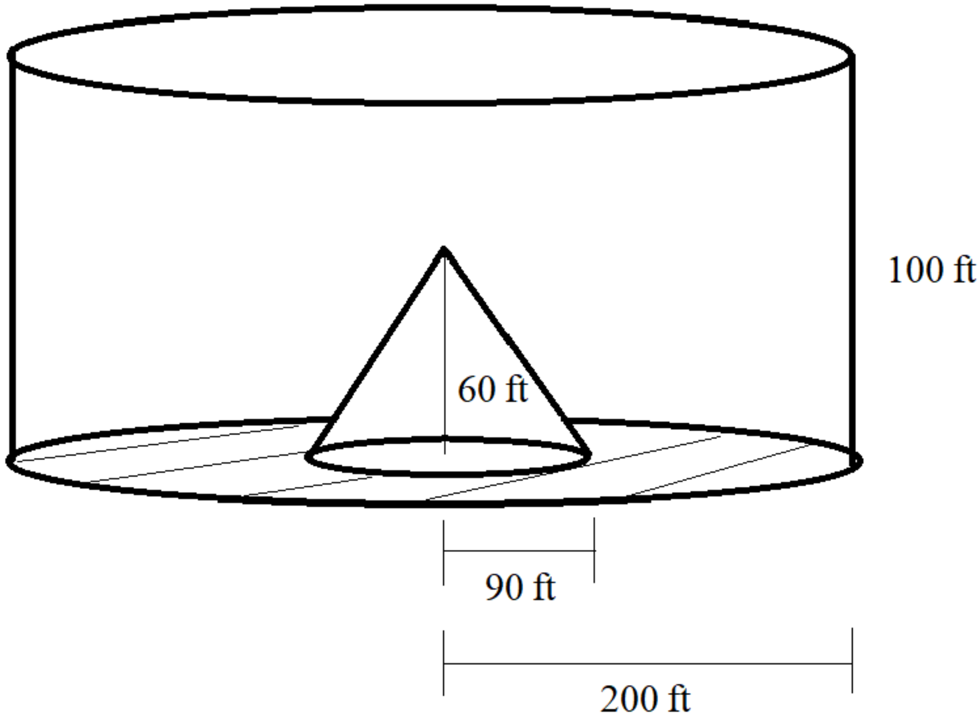

Figure 2: Model of the silo when the pile is 60 feet high.

Determine dA/dh.

$$\frac{dA}{dh} = \frac{9}{2}\pi h$$

dh/dt can be determined by dividing both sides of equation (2) by $(9/4)\pi h^2$.

$$\frac{80,000}{3}\frac{1}{h^2} = \frac{dh}{dt}$$

Hence, we have

$$\begin{aligned}\frac{dA}{dt} &= \frac{9}{2}\pi h \cdot \frac{80,000}{3}\frac{1}{h^2} \\ &= \frac{120,000\pi}{h}\end{aligned}$$

Now evaluate this when $h = 60$.

$$\left.\frac{dA}{dt}\right|_{h=60} = 2,000\pi$$

Therefore, the floor area of the pile is growing at $2,000\pi$ ft^2/h when the pile is 60 feet high.

Part (c)

Using the expression for $t(h)$ in part (a), we can calculate how long it takes for the pile to reach 90 feet.

$$t(90) = \frac{90^3}{80,000} = 9.1125$$

We need a new differential equation for the height after 9.1125 hours since the rate that the volume changes is different.

$$\begin{aligned}\frac{dV}{dt} &= \text{rate in} - \text{rate out}\\ &= 60,000\pi - 20,000\pi\\ &= 40,000\pi\end{aligned}$$

In equation (2), we use $40,000\pi$ on the left side instead of $60,000\pi$.

$$40,000\pi = \frac{9}{4}\pi h^2 \frac{dh}{dt},$$

This is still a differential equation we can solve with separation of variables. Divide both sides by $(9/4)\pi$.

$$\frac{160,000}{9} = h^2 \frac{dh}{dt}$$

Separate variables.

$$\frac{160,000}{9}\,dt = h^2\,dh$$

Integrate both sides.

$$\frac{160,000}{9}t + A = \frac{1}{3}h^3$$

This new expression for the height is valid for $t > 9.1125$, so we determine A using the condition, $h = 90$ when $t = 9.1125$.

$$\frac{160,000}{9} \cdot 9.1125 + A = \frac{1}{3}90^3$$

Solving for A yields $A = 81,000$, so we have

$$\frac{160,000}{9}t + 81,000 = \frac{1}{3}h^3.$$

Solve this equation for t.

$$t(h) = \frac{3(h^3 - 243,000)}{160,000}$$

If we plug in 100 for h, we can find how long it takes for the pile to go from the ground to the top of the silo.

$$t(100) = \frac{3(100^3 - 243,000)}{160,000} \approx 14.2$$

Therefore, it takes $14.2 - 9.1125 \approx 5.1$ hours for the pile to go from 90 feet to 100 feet, the top of the silo.

$$t(h) = \begin{cases} \frac{h^3}{80,000} & 0 < h < 90 \\ \frac{3(h^3 - 243,000)}{160,000} & 90 < h < 100 \end{cases}$$

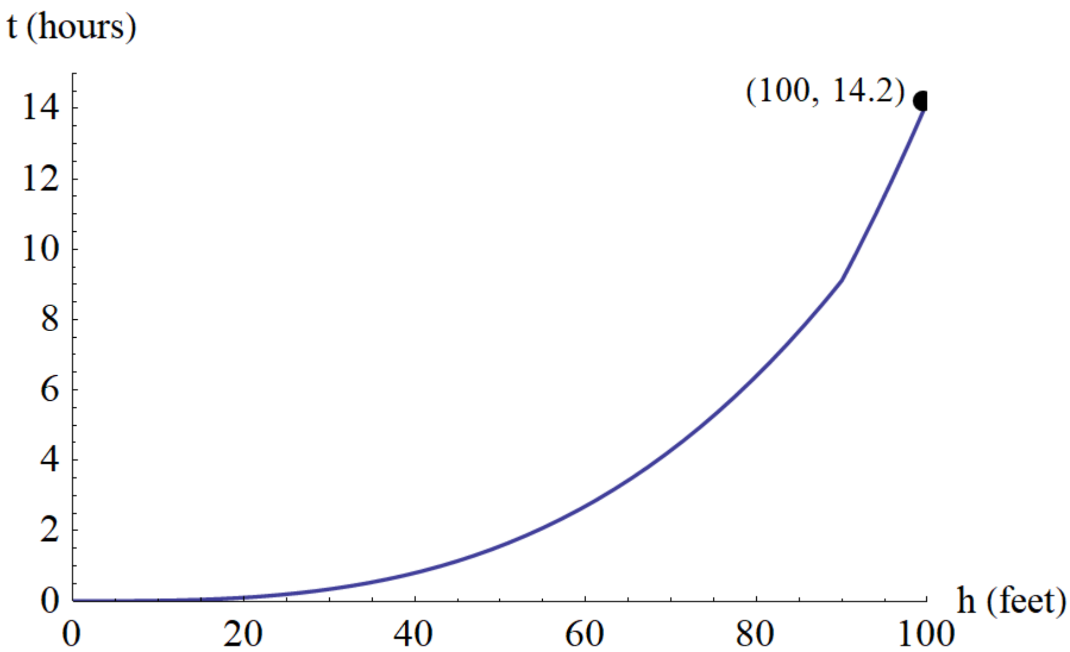

Figure 3: Plot of the time it takes for the pile to get to a certain height when a loader removes ore at a rate of $20,000\pi$ at $h = 90$.